CHARACTERISTIC DAMPING EXPONENTS FOR VIBRATIONS OF MECHANICAL SYSTEMS WITH PARTIAL DISSIPATION

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The effect of partial dissipation on the vibrations of a mechanical system near the equilibrium position, which is stable for certain potential forces, is investigated with the aid of a theorem of E.A.Barbashin and Krasovskii [1].

Results obtained in [2] are extended and made more precise.

The damping coefficients and frequencies of a system with small and large dissipation are calculated approximately.

1. We consider a linear mechanical system near the position of stable equilibrium at an isolated minimum of the potential energy and acted upon by dissipative forces with the dissipation function

$$F = -\frac{1}{2}\sum_{ij=1}^{n} \alpha_{ij} x_i^{\cdot} x_j^{\cdot}$$

Here α_{ij} are constants, F is a negative-definite form of rank k, which is in general less than n, the number of degrees of freedom, i.e. the dissipation is not total.

Let $x_1, \ldots, x_n$ be the normal coordinates. The equations of motion have the form

$$x_i^{\cdot\cdot} + \lambda_i^2 x_i + (\alpha_{i1} x_1^{\cdot} + \ldots + \alpha_{in} x_n^{\cdot}) = 0 \qquad (i = 1, \ldots, n) \tag{1.1}$$

Suppose that none of the numbers $\lambda_1^2, \ldots, \lambda_n^2$ is equal to zero, that there is a group of equal numbers $\lambda_1^2 = \ldots = \lambda_r^2$, and that none of the remaining numbers equals λ_1^2. We note that the variables $x_1, \ldots, x_r$ may undergo any orthogonal transformation, and this transformation will affect only the dissipation coefficients.

Let

$$F_r = -\frac{1}{2}\sum_{ij=1}^{r} \alpha_{ij} x_i^{\cdot} x_j^{\cdot}$$

be that part of the dissipation function which depends only on $x_1^{\cdot}, \ldots, x_r^{\cdot}$. Reducing it to a canonical form by an orthogonal transformation and retaining the old notation for the new variables and coefficients α'_{ij}, we obtain

$$F = -\frac{1}{2}\Big(\sum_{i=1}^{m\leqslant r} \alpha_{ii} x_i^{\cdot 2} + 2\sum_{i>r,\, j\leqslant r} \alpha_{ij} x_i^{\cdot} x_j^{\cdot} + \sum_{ij=r+1}^{n} \alpha_{ij} x_i^{\cdot} x_j^{\cdot}\Big)$$

T h e o r e m 1.1. In order that after addition of the dissipative forces the isolated equilibrium position shall become asymptotically stable with respect to the normal coordinate x_k, belonging to the set $x_1,\ldots,x_k,\ldots,x_r$ of normal coordinates corresponding to the frequency λ_1, it is necessary and sufficient that

$$F_r = -\frac{1}{2}\sum_{i=1}^{m\leqslant r}\alpha_{ii}x_i^{\cdot 2}$$

the part of the dissipation function depending only on the velocities $x^\cdot_1,\ldots, x^\cdot_r$, vanish only for $x^\cdot_k = 0$. In the opposite case, the coordinate x_k remains unaffected by dissipation and will keep vibrating harmonically with frequency λ_1.

In order that the equilibrium state shall be asymptotically stable with respect to all coordinates, it is necessary and sufficient that all functions F_r be sign-definite with respect to all variables. For frequencies λ_i which have the multiplicity of one, this requirement is equivalent to the condition $\alpha_{ii} \neq 0$.

P r o o f . N e c e s s i t y . Let any coefficient in the canonical form F_r be equal to zero. Without loss of generality we will assume that it is α_{11}.

If $\alpha_{11} = 0$, then the variable $x^\cdot_1$ does not appear at all in the dissipation function. Actually, we will assume that there appear in F terms of the form $2\alpha_{s1}x_1^\cdot x_s^\cdot$, where $x^\cdot_s$ is any of the velocities $(s \neq 1)$; then, setting all velocities except $x^\cdot_1$ and $x^\cdot_s$ equal to zero, we obtain

$$-2F = 2\alpha_{s1}x_1^\cdot x_s^\cdot + \alpha_{ss}x_s^{\cdot 2}$$

It is clear that if $\alpha_{s1} \neq 0$, then F may have any sign, which contradicts the assumption that F is negative-definite.

Thus, all α_{s1} are zero and $x^\cdot_1$ will not appear at all in the dissipation function. This means that dissipative terms do not enter into the first equation, and the coordinate x_1 will be unaffected by dissipation. Thus, necessity is proved.

S u f f i c i e n c y From the theorem of Barabashin and Krasovskii [1], applied to Equation

$$\frac{d}{dt}(T-U) = 2F$$

(where T is the kinetic energy and U the force function), we conclude that any perturbed motion will as $t \to \infty$ asymptotically approach either some point on the trajectory of Equations (1.1) which lie entirely within the region $F = 0$, or the origin $(x_i = x_i^\cdot = 0)$. On this trajectory all partial derivatives $\partial F/\partial x_i^\cdot$ and all derivatives with respect to time of these linear forms will of necessity vanish, by virtue of the equations of motion.

Let x_1 correspond to the root λ_1 and let

$$-\frac{\partial F}{\partial x_1^\cdot} = \alpha_{11}x_1^\cdot + u_{21}^\cdot(\lambda_2) + \ldots + u_{n1}^\cdot(\lambda_n)$$

where $u_{21}^\cdot(\lambda_2)$ is a linear form in the velocities $x_j^\cdot$ corresponding to the root λ_2, and so forth. If the system (1.1) admits of solutions, anlong which $F = 0$, then all of these solutions must necessarily lie in the region

$$\frac{\partial F}{\partial x_i^\cdot} = 0, \qquad \frac{d}{dt^2}\left(\frac{\partial F}{\partial x_i^\cdot}\right) = 0$$

where these latter derivatives must be calculated taking into account Equations (1.1), in which we set $dF/dx_1^\cdot = 0$. Calculating the second derivative of $dF/dx_1^\cdot$ using Equation (1.1), we obtain

$$-\frac{d}{dt^2}\left(\frac{\partial F}{\partial x_1^\cdot}\right) = w_1 = \alpha_{11}\lambda_1^2 x_1^\cdot + \lambda_2^2 u_2^\cdot + \ldots + \lambda_n^2 u_n^\cdot = 0$$

Subtracting from the last line $\lambda_2^2\, dF/dx_1^\cdot$, we obtain

$$w_2 = \alpha_{11}(\lambda_1^2 - \lambda_2^2)x_1^\cdot + (\lambda_3^2 - \lambda_2^2)u_3^\cdot + \ldots + (\lambda_n^2 - \lambda_2^2)u_n^\cdot = 0$$

As a result, we find that the form w_2, not containing $u_2^{\cdot}$ and containing $x_1^{\cdot}$, essentially $(\alpha_{11}(\lambda_1^2-\lambda_2^2)\neq 0)$ must vanish.

Differentiating this form twice, by virtue of Equations (1.1), we have

$$w_3=\alpha_{11}\lambda_1^2(\lambda_1^2-\lambda_2^2)x_1^{\cdot}+\lambda_3^2(\lambda_3^2-\lambda_2^2)u_3^{\cdot}+\ldots+\lambda_n^2(\lambda_n^2-\lambda_2^2)u_n^{\cdot}=0$$

Subtracting $\lambda_3^2 w_2$ from it, we obtain

$$w_4=\alpha_{11}(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_2^2)x_1^{\cdot}+(\lambda_4^2-\lambda_3^2)u_4^{\cdot}+\ldots=0$$

Extending this process, we come to the conclusion that $x_1^{\cdot}=0$, which means that $x_1=0$, and so forth. Thus sufficiency is proved.

N o t e . If there are zeros among the numbers $\lambda_1^2,\ldots,\lambda_n^2$ and the equilibrium position is not an isolated one, then reasoning in the same manner, we come to the conclusion that all normal coordinates and velocities corresponding to a nonzero frequency will either vanish with time or oscillate.

For the group of variables corresponding to $\lambda_1=0$, we obtain

$$\alpha_{11}x_1^{\cdot}=\ldots=\alpha_{mm}x_m^{\cdot}=0,\qquad x_1=c_1,\ldots,x_m=c_m$$

If $m=r$, then the motion will asymptotically approach the equilibrium position $x_1=c_1,\ldots,x_m=c_m$; if $m<r$, then $x_{m+1}=v_{0,m+1}t+c_{m+1}$, etc.

It is not difficult to note that if the necessary and sufficient conditions for asymptotic stability of a linear system are fulfilled and none of the frequencies are zero, then asymptotic stability is preserved when any higher-order terms are added. The property mentioned in the note may be violated in a nonlinear system.

2. We assume that the dissipation function contains a small multiplier s and has the form $F=sF_1$, while the dissipative forces become $sdF_1/dx_i^{\cdot}$, where $dF_1/dx_i^{\cdot}$ are comparable in magnitude to the potential forces. Now let $x_1^{\cdot},\ldots,x_r^{\cdot}$ be the variables corresponding to the frequency λ_1. We will seek a root of the characteristic equation of the form

$$\mu_1=\lambda_1 i+a_1 s+b_1 s^2\qquad (i=\sqrt{-1})$$

Substituting it into the characteristic equation of the system (1.1) and introducing the Kronecker symbol δ_{ij}, we obtain

$$\|\delta_{ij}(\mu^2+\lambda_i^2)+\mu s\alpha_{ij}\|=\triangle=0$$

whence we obtain, accurate to order s^{2r}

$$\triangle=\begin{vmatrix}\Lambda_{11} & \ldots & 0 & \alpha_{1r+1}\lambda_1 is & \ldots & \alpha_{1n}\lambda_1 is\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \ldots & \Lambda_{rr} & \alpha_{rr+1}\lambda_1 is & \ldots & \alpha_{rn}\lambda_1 is\\ \alpha_{1r+1}\lambda_1 is & \ldots & \alpha_{rr+1}\lambda_1 is & \lambda_{r+1}^2-\lambda_1^2 & \ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \alpha_{1n}\lambda_1 is & \ldots & \alpha_{rn}\lambda_1 is & 0 & \ldots & \lambda_n^2-\lambda_1^2\end{vmatrix}=0$$

where

$$\Lambda_{jj}=(2a_1+\alpha_{jj})\lambda_1 is+k_1 s^2\quad (j=1,\ldots,r);\qquad (k_1=a_1^2-a_1\alpha_{11}+2b_1\lambda_1 i)$$

Removing the multiplier $\lambda_1 is$ from the first r rows, we obtain

$$\triangle=(\lambda_1 is)^r(2a_1+\alpha_{11})\ldots(2a_1+\alpha_{rr})(\lambda_{r+1}^2-\lambda_1^2)\ldots(\lambda_n^2-\lambda_1^2)+O(s^r)=0$$

whence

$$a_1^1={}^1/_2\,\alpha_{11},\ldots,a_1^r={}^1/_2\,\alpha_{rr}\tag{2.1}$$

If none of the quantities $\alpha_{11},\ldots,\alpha_{rr}$ are equal, then in a similar manner the following equation for determining k_1 is easily obtained:

$$\begin{vmatrix}-\dfrac{k_1}{\lambda_1^2} & \alpha_{1,r+1} & 0\ldots & \alpha_{1n}\\ \alpha_{1,r+1} & \lambda_{r+1}^2-\lambda_1^2 & 0\ldots & 0\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_{1n} & 0 & 0\ldots & \lambda_n^2-\lambda_1^2\end{vmatrix}=0$$

Hence we obtain $$\frac{k_1}{\lambda_1^2} = -\frac{\alpha_{1,\,r+1}^2}{\lambda_{r+1}^2 - \lambda_1^2} - \ldots - \frac{\alpha_{1n}^2}{\lambda_n^2 - \lambda_1^2}$$

But on the other hand

$$k_1 = a_1^2 - a_1\alpha_{11} + 2b_1\lambda_1 i = -{}^1/_4\,\alpha_{11}^2 + 2b_1\lambda_1 i$$

Solving for b_1, we have

$$b_1^{(1,2)} = \pm\frac{i}{2\lambda_1}\left(\frac{\alpha_{11}^2}{4} - \lambda_1^2 \sum_{j=r+1}^{n} \frac{\alpha_{1j}^2}{\lambda_j^2 - \lambda_1^2}\right)$$

and similarly

$$b_2^{(1,2)} = \pm\frac{i}{2\lambda_1}\left(\frac{\alpha_{22}^2}{4} - \lambda_1^2 \sum_{j=r+1}^{n} \frac{\alpha_{2j}^2}{\lambda_j^2 - \lambda_1^2}\right) \qquad \text{etc.}$$

As a result, we come to the conclusion that if none of the coefficients of the canonical quadratic form F_r corresponding to the variables $x_1^\cdot, \ldots$ $\ldots, x_r^\cdot$ and frequency λ_1 are equal, then the characteristic exponent takes the form

$$\mu_j = -\frac{1}{2}\alpha_{jj}s \pm i\left[\lambda_1 + \frac{s^2}{2\lambda_1}\left(\frac{\alpha_{jj}^2}{4} - \sum_{i=r+1}^{n} \frac{\alpha_{ji}^2}{\lambda_i^2 - \lambda_1^2}\right)\right]$$

accurate to terms of order s^3.

3. We shall now consider the effect of large dissipation with a dissipation function F of the form

$$F = \frac{1}{s}F_1 = -\frac{1}{2s}\sum_{ij=1}^{n} \alpha_{ij}x_i^\cdot x_j^\cdot = -\frac{1}{2s}\sum_{ij=1}^{k<n} \beta_{ij}z^\cdot_i z^\cdot_j$$

Here s is a small multiplier, $k < n$ is the rank of the form F, which is negative definite with respect to $z_1^\cdot, \ldots, z_k^\cdot$, the linear forms of the original velocities. We shall also assume that F makes the initial equilibrium position asymptotically stable. Let $z_1^\cdot, \ldots, z_k^\cdot$ be expressed in terms of the canonical momenta of the system by

$$z_i^\cdot = \theta_{i1}\frac{\partial T}{\partial x_1^\cdot} + \ldots + \theta_{in}\frac{\partial T}{\partial x_n^\cdot} \qquad (i = 1, \ldots, k) \tag{3.1}$$

Consider the change of variables

$$x_j = \theta_{1j}y_1 + \ldots + \theta_{kj}y_k + \ldots + \theta_{nj}y_n \tag{3.2}$$

where $\theta_{i1}, \ldots, \theta_{in}\,(i \leqslant k)$ are taken from Formula (3.1), while the remaining θ_{ij} are arbitrary and are related only by the condition that the transformation (3.2) be nonsingular. In the new variables we have the equality

$$\frac{\partial T}{\partial y_i^\cdot} = \theta_{i1}\frac{\partial T}{\partial x_1^\cdot} + \ldots + \theta_{in}\frac{\partial T}{\partial x_n^\cdot} = z_i^\cdot \qquad (i = 1, \ldots, k)$$

Replacing $y_1^\cdot, \ldots, y_k^\cdot$ by $z_1^\cdot, \ldots, z_k^\cdot$ according to Routh's theorem [3], we find that the kinetic energy takes the form

$$T = T_1(z_1^\cdot, \ldots, z_k^\cdot) + T_2(y_{k+1}^\cdot, \ldots, y_n^\cdot)$$

in the variables $z_1^\cdot, \ldots, z_k^\cdot$, $y_{k+1}, \ldots, y_n^\cdot$ and does not contain products $y_i^\cdot z_j^\cdot$. Both quadratic forms T_1 and T_2 will be positive definite functions of the variables appearing in them. Therefore the forms T_1 and F may, by a simultaneous transformation, be reduced to a sum of squares

$$2T_1 = z_1^{\cdot 2} + \ldots + z_k^{\cdot 2}, \quad -2sF = a_{11}z_1^{\cdot 2} + \ldots + a_{kk}z_k^2$$

Let the force function U take the form

$$U = U_1(z_1, \ldots, z_k) + U_2(z_j, y_i) + U_3(y_{k+1}, \ldots, y_n)$$

in the variables $z_1, \ldots, z_k, y_{k+1}, \ldots, y_n$, where U_1 depends only on the variables $z_1, \ldots, z_k$; U_2 contains only the products $z_j y_i$, and U_3 only the variables $y_{k+1}, \ldots, y_r$. The term U_3 is a negative definite quadratic form of its variables, which may be simultaneously reduced with T_2 by an orthogonal transformation to a sum of squares

$$2T_2 = y_{k+1}^{\cdot 2} + \ldots + y_n^{\cdot 2}, \; 2U_3 = -\nu_{k+1}^2 y_{k+1}^2 - \ldots - \nu_n^2 y_n^2$$

Retaining the old notation for the new variables, we remember, however, that all the derivations which have been carried out refer to the coefficients of the system of equations in which the kinetic energy T, the potential function U and the dissipation function F have, with the help of a linear substitution, been written in the special form

$$2T = z_1^{\cdot 2} + \ldots + z_k^{\cdot 2} + y_{k+1}^{\cdot 2} + \ldots + y_n^{\cdot 2}$$

$$2U = -\sum_{ij=1}^{k} c_{ij} z_i z_j - 2 \sum_{i>k,\; j\leqslant k} c_{ij} z_j y_i - \nu_{k+1}^2 y_{k+1}^2 - \ldots - \nu_n^2 y_n^2 \qquad (3.3)$$

$$-2sF = a_{11} z_1^{\cdot 2} + \ldots + a_{kk} z_k^{\cdot 2}$$

Before proceeding with the investigation, we will prove an auxiliary lemma.

L e m m a 3.1. If the introduction of dissipation makes the equilibrium position asymptotically stable, then none of the coefficients $\nu_{k+1}^2, \ldots, \nu_n^2$ may be equal.

P r o o f . Setting $z_1 = \ldots = z_k = 0$ in Equations (3.3), we obtain

$$2T^\circ = y_{k+1}^{\cdot 2} + \ldots + y_n^{\cdot 2}, \qquad 2U^\circ = -\nu_{k+1}^2 y_{k+1}^2 - \ldots - \nu_n^2 y_n^2$$

Hence it is clear $\nu_{k+1}, \ldots, \nu_n$ are the principal frequencies of the system which is obtained from the original system after introduction of the additional constraints $z_1 = \ldots = z_k = 0$.

Let $x_1, \ldots, x_n$ be the normal coordinates of the original system, and $z_1, \ldots, z_k$ be expressed in terms of $x_1, \ldots, x_n$ by

$$z_i = \gamma_{i1} x_1 + \ldots + \gamma_{in} x_n$$

Differentiating this equation once with respect to time, we obtain

$$z_i^\cdot = \gamma_{i1} x_1^\cdot + \ldots + \gamma_{in} x_n^\cdot \qquad (i = 1, \ldots, k)$$

The dissipation function vanishes for $z_1^\cdot = \ldots = z_k^\cdot = 0$, consequently this condition is equivalent to the system

$$-2s \frac{\partial F}{\partial x_1^\cdot} = \alpha_{11} x_1^\cdot + u_{21}^\cdot(\lambda_2) + \ldots + u_{n1}^\cdot(\lambda_n) = 0$$

$$-2s \frac{\partial F}{\partial x_n^\cdot} = u_{1n}^\cdot(\lambda_1) + \ldots + u_{nn}^\cdot(\lambda_n) = 0 \qquad (3.4)$$

The forms $u_{ij}^\cdot$ have already been encountered in Section 1. That part of the first row which depends on the velocities corresponding to the frequecy λ_2 is denoted by $u_{21}^\cdot(\lambda_2)$, while $u_{31}^\cdot(\lambda_2)$ denotes the part depending on the root λ_3 , and so forth.

Since $z_1^\cdot = \ldots = z_k^\cdot = 0$ is equivalent to the system (3.4), then $z_1 = \ldots$ $\ldots = z_k = 0$ will be equivalent to the system

$$\alpha_{11} x_1 + u_{21}(\lambda_2) + \ldots + u_{n1}(\lambda_n) = 0$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$u_{1n}(\lambda_1) + \ldots + u_{nn}(\lambda_n) = 0$$

which is obtained from (3.4) by replacing the velocities $x_1^\cdot, \ldots, x_n^\cdot$ by the coordinates $x_1, \ldots, x_n$. Imposing the constraint

$$v_1 = \alpha_{11} x_1 + u_{21}(\lambda_2) + \ldots + u_{n1}(\lambda_w) = 0$$

on the mechanical system, we find, by virtue of a well-known theorem [4], that after the constraint $v_1 = 0$ has been applied, the frequencies of the constrained system will either partition the frequencies of the original system, or, in the case of a multiple frequency, part of them will be conserved. Thus, if the sum of coefficients of the form $u_{2}(\lambda_2)$, etc. differs from zero, then the multiplicity of the root will be decreased at least by 1. Because of this property, it is clear that if the frequency λ_k is preserved in the new system, then its multiplicity will either be decreased or remain unchanged, and the cases where the multiplicity of even one of the preserved frequencies is increased, or where two or several of the newly appearing frequencies coincide, are impossible.

Since the conditions of Lemma assume asymptotic stability, then from the theorem of Section 1, it necessarily follows that $\alpha_{11} > 0$, since $-2sF_r = \alpha_{11}x_1^{\cdot 2} + \ldots + \alpha_{rr}x_r^{\cdot 2}$ is positive definite. Consequently, afer applying the constraint $v_1 = 0$, the multiplicity of the frequency λ_1 is lowered not less than 1. Since $v_1 = 0$ does not affect the coordinates $x_2, \ldots, x_r$, then it is clear that the system will have a frequency λ_1 of multiplicity $r-1$, hence the lowering of the multiplicity of the root λ_1 is exactly equal to 1. Imposing the constraint $v_2 = 0$, and noting that the equation $v_2 = 0$ does not contain $x_1, x_3, \ldots, x_r$, but only x_2 with the coefficient $\alpha_{22} > 0$, we again obtain a reduction of the multiplicity by 1, and so forth.

In the end, the system loses the root λ_1. In exactly the same way we prove that it necessarily loses all roots (in the sense that all the frequencies will be new ones), and we conclude that in a system with the additional constraints $z_1 = \ldots = z_k = 0$ there are no multiple principal frequancies, and consequently $\nu_{k+1}, \ldots, \nu_n$ are all different.

We now write the equations of motion

$$z_i^{\cdot\cdot} + s^{-1}a_{ii}z_i^{\cdot} + c_{i1}z_1 + \ldots + c_{ik}z_k + c_{ik+1}y_{k+1} + \ldots + c_{in}y_n = 0 \qquad (i = 1, \ldots, k)$$
$$y_j^{\cdot\cdot} + \nu_j^2 y_j + c_{j1}z_1 + \ldots + c_{jk}z_k = 0 \qquad (j = k+1, \ldots, n)$$

The characteristic equation has the form

$$\frac{1}{s^r}\begin{vmatrix} M_{11} \ldots sc_{1k} & sc_{1k+1} \ldots sc_{1n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ sc_{1k} \ldots M_{kk} & sc_{k,k+1} \ldots sc_{kn} \\ c_{1,k+1} \ldots c_{k,k+1} & \mu^2 + \nu_{k+1}^2 \ldots 0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ c_{1n} \ldots c_{kn} & 0 \ldots \mu^2 + \nu_n^2 \end{vmatrix} = 0$$

$$M_{ii} = s\mu^2 + a_{ij}\mu + sc_{ii} \quad (i = 1, 2, \ldots, k)$$

Seeking roots of this equation of the form

$$\mu_n = \pm \nu_n i + a_{1n}s + a_{2n}s^2 + O(s^2)$$

we obtain the equation for determining a_{1n}

$$\begin{vmatrix} \nu_n i a_{11} & 0 & \ldots & 0 & 0 & \ldots & 0 & c_{1n} \\ \ldots & & & & & & & \ldots \\ 0 & 0 & \ldots & \nu_n i a_{kk} & 0 & \ldots & 0 & c_{kn} \\ c_{k+1,1} & c_{k+1,2} & \ldots & c_{k+1,k} & \nu_{k+1}^2 - \nu_n^2 & \ldots & 0 & c_{k+1,n} \\ \ldots & & & & & & & \ldots \\ c_{n-1,1} & c_{n-1,2} & \ldots & c_{n-1,k} & 0 & \ldots & \nu_{n-1}^2 - \nu_n^2 & c_{n-1,n} \\ c_{n1} & c_{n2} & \ldots & c_{nk} & 0 & \ldots & 0 & 2\nu_n i a_{n1} \end{vmatrix} = 0$$

In the columns numbered from $k+1$ to $n-1$, inclusively, there is only a single nonzero element $\nu_{k+1}^2 - \nu_n^2$, etc., hence these columns may be deleted, together with the corresponding rows. We now multiply the last row by $1/\nu_n i$ and then subtract the first column, multiplied by $c_{1n}/\nu_n a_{11} i$, from the last, we multiply the second column by $c_{2n}/\nu_n a_{11} i$ and subtract from the last, and so forth. As a result we obtain

$$a_{1n} = -\frac{1}{2\nu_n^2}\left(\frac{c_{1n}^2}{a_{11}} + \frac{c_{2n}^2}{a_{22}} + \ldots + \frac{c_{kn}^2}{a_{kk}}\right)$$

and similarly

$$a_{1j} = -\frac{1}{2\nu_j^2}\left(\frac{c_{ij}^2}{a_{11}} + \ldots + \frac{c_{kj}^2}{a_{kk}}\right)$$

The calculation of a_{2j} (the coefficient of s^2 in the expansion of the pair of roots numbered $k+1, \ldots, n$) shows that all of them will be purely imaginary. Their formulas are, however, rather cumbersome and will not be cited. The remaining pairs of roots of the characteristic equation will be sought in the form

$$\mu_j = b_{1j} / s + b_{2j}s + O(s^2)$$

Substituting μ_j into the equation, we easily obtain

$$b_{1j}^1 = -a_{jj}, \qquad b_{1j}^2 = 0$$

To find b_{2j} we revert to the original variables $x_1, \ldots, x_n$ and reduce the force and dissipation functions

$$2U = -\sum_{i=1}^{n} \lambda_i x_i^2, \qquad -2Fs = \sum_{i=1}^{n} \alpha_{ii} x_i^{\boldsymbol\cdot} x_j^{\boldsymbol\cdot}$$

by a simultaneous orthogonal transformation to the form

$$2U = -\sum_{i=1}^{n} \lambda_i' x_i'^2, \qquad -2Fs = \sum_{i=1}^{n} \alpha_{ii}' x_i^{\boldsymbol\cdot\prime 2}$$

Let the kinetic energy T take the form

$$2T = \sum A_{ij} x_i^{\boldsymbol\cdot\prime} x_j^{\boldsymbol\cdot\prime}$$

in these variables.

After these transformations the characteristic equation of the system is written in the form

$$\| \mu^2 A_{ij} + \delta_{ij}\lambda_i'^2 + \delta_{ij}\mu / s \| = 0$$

Here δ_{ij} is the usual Kronecker symbol, and $\delta_{ij}' = \delta_{ij}$ if $i, j \leqslant k$ and equals zero for either i or j greater than k. Thus we obtain

$$b^2_j = -\lambda_j'^2$$

As a result, it is clear that for large dissipative forces there will always exist in the system solutions with small damping, and if the dissipation is partial with rank $k < h$, but it gives the system asymptotic stability, then there exist $2(n - k)$ oscillatory solutions with exponents

$$\mu_i = \pm \nu_j i - \frac{s}{2\nu_j^2}\left(\frac{c_{1j}^2}{a_{11}} + \ldots + \frac{c_{kj}^2}{a_{kk}}\right) + O(s) \qquad (j = k+1, \ldots, n)$$

and $2k$ with exponents

$$\mu_{j1} \approx -a_{jj} / s, \quad \mu_{j_2} = -\lambda_j'^2 s + O(s)$$

The first group of these exponents corresponds to strong damping.

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